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Drazin inverse of partitioned matrices in terms of Banachiewicz–Schur forms ☆

N. Castro-González *, M.F. Martínez-Serrano

Departamento de Matemática Aplicada, Facultad de Informática, Universidad Politécnica de Madrid, 28660 Boadilla del Monte, Madrid, Spain

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ABSTRACT

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a partitioned matrix, where A and D are square matrices. Denote the Drazin inverse of A by A^D. The purpose of this paper is twofold. Firstly, we develop conditions under which the Drazin inverse of M having generalized Schur complement, $S = D - CA^DB$, group invertible, can be expressed in terms of a matrix in the Banachiewicz–Schur form and its powers. Secondly, we deal with partitioned matrices satisfying $\text{rank}(M) = \text{rank}(A^D) + \text{rank}(S^D)$, and give conditions under which the group inverse of M exists and a formula for its computation.

1. Introduction

Let $\mathcal{C}_{m,n}$ be the set of $m \times n$ complex matrices. Let us recall that a matrix $A^- \in \mathcal{C}_{n,m}$ is an inner inverse of a given matrix $A \in \mathcal{C}_{m,n}$ if $AA^-A = A$. Also, a matrix $X \in \mathcal{C}_{n,m}$ is a generalized reflexive inverse of A (inner and outer inverse) if $AXA = A$ and $XAX = X$.

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* Corresponding author.
E-mail addresses: nieves@fi.upm.es (N. Castro-González), fmartinez@fi.upm.es (M.F. Martínez-Serrano).

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Now, let $A \in \mathcal{C}_{m,m}$. Let us define the index of A, $\operatorname{ind}(A)$, as the smallest non-negative integer r such that $\operatorname{rank}(A^r) = \operatorname{rank}(A^{r+1})$. If $\operatorname{ind}(A) = r$, the Drazin inverse of A is the unique matrix $X \in \mathcal{C}_{m,m}$ satisfying the relations

$$XAX = X, \quad AX = XA, \quad A^{k+1}X = A^k \quad \text{for all } k \geqslant r. \tag{1.1}$$

The Drazin inverse of A will be specified by A^D.

If $\operatorname{ind}(A) = 0$, then A is nonsingular and the solution to (1.1) is $A^D = A^{-1}$. If $\operatorname{ind}(A) = 1$, then A^D is a generalized reflexive inverse of A, and it is called the group inverse of A, denoted by $A^\sharp$.

We will denote by A^π the eigenprojection of A corresponding to the eigenvalue 0, which is given by $A^\pi = I - AA^D$.

For the theory of generalized inverses and its applications, we refer the reader to [2,4]. In the following proposition, we have compiled some basic facts about the Drazin inverse, which will be used throughout the paper. We will denote by $\mathcal{R}(A)$ and $\mathcal{N}(A)$ the range and the null space of A, respectively.

Proposition 1.1. *Let $A, B \in \mathcal{C}_{n,n}$ with $\operatorname{ind}(A) = r$ and let $K = A^2A^D$. Then*

(a) $\mathcal{R}(A^D) = \mathcal{R}(A^r)$ *and* $\mathcal{N}(A^D) = \mathcal{N}(A^r)$.
(b) $\mathcal{R}(A^\pi) = \mathcal{N}(A^r)$ *and* $\mathcal{N}(A^\pi) = \mathcal{R}(A^r)$.
(c) $\operatorname{ind}(K) = 1$, $K^D = A^D$, $\mathcal{R}(K) = \mathcal{R}(A^D)$ *and* $\mathcal{N}(K) = \mathcal{N}(A^D)$.
(d) *If P is nonsingular and $B = PAP^{-1}$, then $B^D = PA^DP^{-1}$.*
(e) *If $r > 0$, then there exists a nonsingular matrix P such that we can write A in the core-nilpotent block form*

$$A = P\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}P^{-1}, \quad A_1 \in \mathcal{C}_{k,k} \text{ nonsingular}, \ k = \operatorname{rank}(A^r), \ A_2^r = 0, \tag{1.2}$$

and, relative to this form, we have

$$A^D = P\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}, \quad A^\pi = P\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}P^{-1}.$$

Moreover, if $\operatorname{ind}(A) = 1$, then $A_2 = 0$ in (1.2) and so $A^\pi A = AA^\pi = 0$.

Throughout this paper we consider $M \in \mathcal{C}_{m+n,m+n}$ partitioned as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A \in \mathcal{C}_{m,m}, \ D \in \mathcal{C}_{n,n}. \tag{1.3}$$

Representations for the Drazin inverse for block matrices were given in the literature under certain conditions [3,6,7,10,17]. In recent papers [1,15] necessary and sufficient conditions were derived for a partitioned matrix to have several generalized inverses, including inner, reflexive and Moore–Penrose inverse, with Banachiewicz–Schur form. We recall that, if $A^=$ denotes a generalized inverse of A, then the generalized Schur complement of A in M is defined as $S = D - CA^=B$, and we say that the generalized inverse of M has the Banachiewicz–Schur form when it is expressible in the form

$$M^= = \begin{pmatrix} A^= + A^=BS^=CA^= & -A^=BS^= \\ -S^=CA^= & S^= \end{pmatrix}.$$

The Schur complement is a basic tool in many areas of matrix analysis [18].

In the sequel we consider the Drazin inverse of A and the generalized Schur complement

$$S = D - CA^DB. \tag{1.4}$$

Under the condition that S is nonsingular, the following result is known [16].

Lemma 1.2. *Let M be partitioned as in (1.3) and let S be as in (1.4). If S is nonsingular, $A^\pi B = 0$ and $CA^\pi = 0$, then*

$$M^D = \begin{pmatrix} A^D + A^D BS^{-1}CA^D & -A^D BS^{-1} \\ -S^{-1}CA^D & S^{-1} \end{pmatrix}. \tag{1.5}$$

The case in which the generalized Schur complement is equal to zero has been studied in the literature. In [4, Lemma 3.3.1 and Theorem 7.7.7], it was proved that if A is nonsingular, then $\operatorname{rank}(M) = \operatorname{rank}(A)$ if and only if $S = 0$. In this situation, the group inverse of M exists if and only if $I + A^{-1}BCA^{-1}$ is nonsingular, in which case

$$M^\sharp = \begin{pmatrix} I \\ CA^{-1} \end{pmatrix} (WAW)^{-1} \begin{pmatrix} I & A^{-1}B \end{pmatrix}, \quad W = I + A^{-1}BCA^{-1}. \tag{1.6}$$

This representation was very useful for the study of the perturbation of the Drazin inverse in [5].

In [12], under the assumptions that $A^\pi B = 0$, $CA^\pi = 0$ and $S = 0$, it was shown that the Drazin inverse of M is expressible in the form

$$M^D = \begin{pmatrix} I \\ CA^D \end{pmatrix} A \left[(WA)^D\right]^2 \begin{pmatrix} I & A^D B \end{pmatrix}, \quad W = AA^D + A^D BCA^D. \tag{1.7}$$

Hartwig et al. in [8, Theorems 3.1 and 4.1] gave expressions for the Drazin inverse of M in the cases when S is nonsingular and $S = 0$, under conditions $CA^\pi B = 0$ and $AA^\pi B = 0$. The formulas showed therein involve a matrix in the form (1.5) for the case when S is nonsingular and a matrix in the form (1.7) for the case when $S = 0$.

The paper is organized as follows. In Section 2, we develop conditions under which the Drazin inverse of a partitioned matrix as in (1.3) having group invertible generalized Schur complement, can be expressed in terms of the Banachiewicz-Schur form of M and its powers. In Section 3, we deal with partitioned matrix satisfying the rank formula $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$, and we give conditions under which the group inverse of M exists and a formula for its computation. We emphasize that when the rank formula holds, then S will be group invertible. From our results we derive the characterization of the group invertibility of M for the cases in which A is nonsingular, and S is nonsingular.

In our development we will use the following lemma. It is well-known that if $AB = BA = 0$, then $(A+B)^D = A^D + B^D$. An extension of this additive result under the one side condition $AB = 0$ was given in [9].

Lemma 1.3. *Let $A \in \mathcal{C}_{n,n}$ and let $B \in \mathcal{C}_{n,n}$ be nilpotent of index r.*

(i) *If $BA = 0$, then $(A+B)^D = \sum_{i=0}^{r-1} (A^D)^{i+1} B^i$.*
(ii) *If $AB = 0$, then $(A+B)^D = \sum_{i=0}^{r-1} B^i (A^D)^{i+1}$.*

In order to establish rank equalities for block matrices, we need some rank formulas [11].

Lemma 1.4. *Let $A \in \mathcal{C}_{m,p}$, $B \in \mathcal{C}_{m,k}$, $C \in \mathcal{C}_{n,p}$, $D \in \mathcal{C}_{n,k}$, $G \in \mathcal{C}_{p,k}$. Then*

(i) $\operatorname{rank}(AG) = \operatorname{rank}(G) - \dim(\mathcal{R}(G) \cap \mathcal{N}(A))$.
(ii) $\operatorname{rank}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \operatorname{rank}(B) + \operatorname{rank}(C) + \operatorname{rank}[(I_m - BB^-)A(I_p - C^-C)]$.
(iii) $\operatorname{rank}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{rank}(A) + \operatorname{rank}\begin{pmatrix} 0 & (I_m - AA^-)B \\ C(I_p - A^-A) & D - CA^-B \end{pmatrix}$.

2. Drazin inverse of block matrices in terms of Banachiewicz–Schur forms

In this section we address the problem of developing conditions under which the Drazin inverse of a partitioned matrix as in (1.3) can be obtained by a formula which involves the Banachiewicz–Schur form. First, we will derive a formula under conditions $A^\pi B = 0$ and $S^\pi C = 0$, with A and S being group invertible. Secondly, we will extend this result under less restrictive conditions. In particular we recover the case $AA^\pi B = 0$, $CA^\pi B = 0$ and $S^\pi C = 0$.

We will use the notation of the following lemma.

Lemma 2.1. *Let M be partitioned as in (1.3) and let S be as in (1.4). Let us introduce the nonsingular matrix*

$$U = \begin{pmatrix} I & A^D B \\ S^D C & I + S^D CA^D B \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} I + A^D BS^D C & -A^D B \\ -S^D C & I \end{pmatrix}. \tag{2.1}$$

If $CA^\pi B = 0$, then

$$UMU^{-1} = \begin{pmatrix} A + A^D BS^\pi C - A^\pi BS^D C & A^D BS + A^\pi B \\ S^D C(A + A^D BS^\pi C) + S^\pi C & S + S^D CA^D BS \end{pmatrix}. \tag{2.2}$$

The following theorem is an extension of [13, Theorem 3.3].

Theorem 2.2. *Let M be partitioned as in (1.3) and let S be as in (1.4). If $\operatorname{ind}(A) = 1$, $\operatorname{ind}(S) \leqslant 1$, $A^\pi B = 0$ and $S^\pi C = 0$, then*

$$M^D = \begin{pmatrix} A^\sharp + A^\sharp BS^\sharp CA^\sharp & -A^\sharp BS^\sharp \\ -S^\sharp CA^\sharp & S^\sharp \end{pmatrix} \begin{pmatrix} I - A^\sharp BS^\sharp CA^\pi & A^\sharp BS^\pi \\ S^\sharp CA^\pi & I \end{pmatrix}. \tag{2.3}$$

Proof. First, we observe that if $\operatorname{ind}(S) = 0$, then S is nonsingular. In this case, the expression (2.3) can be derived from [8, Theorem 3.1].

In the sequel, we assume that $\operatorname{ind}(S) = 1$. Under assumptions of this theorem, the expression (2.2) reduces to

$$UMU^{-1} = \begin{pmatrix} A & A^\sharp BS \\ S^\sharp CA & S + S^\sharp CA^\sharp BS \end{pmatrix} := \widetilde{M}, \tag{2.4}$$

where U and U^{-1} are defined as in (2.1). By Proposition 1.1 (e), we have $S = Q\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}Q^{-1}$, where S_1 is nonsingular. Now write, relative to the above decomposition, $BQ = (B_1\ B_2)$ and $Q^{-1}C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$. Then

$$\begin{pmatrix} I & 0 \\ 0 & Q^{-1} \end{pmatrix} \widetilde{M} \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} A & A^\sharp B_1 S_1 & 0 \\ S_1^{-1} C_1 A & S_1 + S_1^{-1} C_1 A^\sharp B_1 S_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Schur complement of A in the 2×2 block submatrix of the matrix on the right hand side is equal to S_1 and, thus, it is nonsingular. Since $A^\pi A^\sharp = 0$ and $AA^\pi = 0$ we can apply Lemma 1.2 to obtain the Drazin inverse of the 2×2 block submatrix and, consequently, we get

$$\begin{aligned} \widetilde{M}^D &= \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} A^\sharp + \left(A^\sharp\right)^2 B_1 S_1^{-1} C_1 AA^\sharp & -\left(A^\sharp\right)^2 B_1 & 0 \\ -\left(S_1^{-1}\right)^2 C_1 AA^\sharp & S_1^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q^{-1} \end{pmatrix} \\ &= \begin{pmatrix} A^\sharp + \left(A^\sharp\right)^2 BS^\sharp CAA^\sharp & -\left(A^\sharp\right)^2 BSS^\sharp \\ -\left(S^\sharp\right)^2 CAA^\sharp & S^\sharp \end{pmatrix}. \end{aligned}$$

Finally, in view of (2.4) and on account of Proposition 1.1 (d), we compute

$$\widetilde{M}^D U = \begin{pmatrix} A^\sharp + \left(A^\sharp\right)^2 BS^\sharp CAA^\sharp & -\left(A^\sharp\right)^2 BSS^\sharp \\ -\left(S^\sharp\right)^2 CAA^\sharp & S^\sharp \end{pmatrix} \begin{pmatrix} I & A^\sharp B \\ S^\sharp C & I + S^\sharp CA^\sharp B \end{pmatrix}$$

$$= \begin{pmatrix} A^\sharp - (A^\sharp)^2BS^\sharp CA^\pi & (A^\sharp)^2BS^\pi \\ (S^\sharp)^2CA^\pi & S^\sharp \end{pmatrix}$$

and

$$M^D = U^{-1}\widetilde{M}^DU = \begin{pmatrix} I + A^\sharp BS^\sharp C & -A^\sharp B \\ -S^\sharp C & I \end{pmatrix} \begin{pmatrix} A^\sharp - (A^\sharp)^2BS^\sharp CA^\pi & (A^\sharp)^2BS^\pi \\ (S^\sharp)^2CA^\pi & S^\sharp \end{pmatrix},$$

which gives us the desired result (2.3). □

The following particular case recovers [13, Theorem 3.2].

Corollary 2.3. *Let M be partitioned as in (1.3) and let S be as in (1.4). If* $\operatorname{ind}(A) = 1$, $\operatorname{ind}(S) \leqslant 1$, $A^\pi B = 0$, $CA^\pi = 0$, $S^\pi C = 0$ *and* $BS^\pi = 0$, *then* $\operatorname{ind}(M) = 1$ *and*

$$M^\sharp = \begin{pmatrix} A^\sharp + A^\sharp BS^\sharp CA^\sharp & -A^\sharp BS^\sharp \\ -S^\sharp CA^\sharp & S^\sharp \end{pmatrix}.$$

Using the following lemma we can prove the counterpart of Theorem 2.2.

Lemma 2.4. *Let M be partitioned as in (1.3) and let S be as in (1.4). Let us introduce the nonsingular matrix*

$$V = \begin{pmatrix} I & BS^D \\ CA^D & I + CA^DBS^D \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} I + BS^DCA^D & -BS^D \\ -CA^D & I \end{pmatrix}. \tag{2.5}$$

If $CA^\pi B = 0$, *then*

$$V^{-1}MV = \begin{pmatrix} A + BS^\pi CA^D - BS^DCA^\pi & (A + BS^\pi CA^D)BS^D + BS^\pi \\ SCA^D + CA^\pi & S + SCA^DBS^D \end{pmatrix}.$$

Theorem 2.5. *Let M be partitioned as in (1.3) and let S be as in (1.4). If* $\operatorname{ind}(A) = 1$, $\operatorname{ind}(S) \leqslant 1$, $CA^\pi = 0$ *and* $BS^\pi = 0$, *then*

$$M^D = \begin{pmatrix} I - A^\pi BS^\sharp CA^\sharp & A^\pi BS^\sharp \\ S^\pi CA^\sharp & I \end{pmatrix} \begin{pmatrix} A^\sharp + A^\sharp BS^\sharp CA^\sharp & -A^\sharp BS^\sharp \\ -S^\sharp CA^\sharp & S^\sharp \end{pmatrix}. \tag{2.6}$$

Now, we state the main result of this section. The significance of the assumptions in the theorem will be shown in the next section.

Theorem 2.6. *Let M be partitioned as in (1.3) and let S be as in (1.4) with* $\operatorname{ind}(S) \leqslant 1$. *If* $AA^\pi B = 0$, $CA^\pi B = 0$, $S^\pi CA^\pi = 0$ *and* $W = I + A^DBS^\pi CA^D$ *is nonsingular, then*

$$\begin{aligned} M^D = {} & \left[I + \begin{pmatrix} -A^DBS^\pi C & (A^\pi + (I - W))B \\ S^\pi C & S^\pi CA^DB \end{pmatrix} R + \begin{pmatrix} A^\pi BS^\pi C & A^\pi BS^\pi CA^DB \\ 0 & 0 \end{pmatrix} R^2 \right] \\ & \times R \left[\begin{pmatrix} I & A^DBS^\pi \\ 0 & I \end{pmatrix} + \sum_{i=0}^{r-1} R^{i+1} \begin{pmatrix} 0 & 0 \\ CA^\pi A^i & 0 \end{pmatrix} \right], \end{aligned} \tag{2.7}$$

where $r = \operatorname{ind}(A)$ *and*

$$R = \begin{pmatrix} A^D + A^DBS^\sharp CA^D & -A^DBS^\sharp \\ -S^\sharp CA^D & S^\sharp \end{pmatrix} \begin{pmatrix} W^{-1} & 0 \\ 0 & I \end{pmatrix}. \tag{2.8}$$

Proof. With the notation $W = I + A^D BS^\pi CA^D$, using $CA^\pi B = 0$ and $S^\pi CA^\pi = 0$, from (2.2) we obtain

$$\begin{aligned} UMU^{-1} &= \begin{pmatrix} WA^2A^D & A^DBS \\ S^\sharp CWA^2A^D & S + S^\sharp CA^DBS \end{pmatrix} + \begin{pmatrix} AA^\pi & 0 \\ S^\sharp CAA^\pi & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} -A^\pi BS^\sharp C & A^\pi B \\ S^\pi C & 0 \end{pmatrix} := X + Y + Z. \end{aligned} \tag{2.9}$$

Under the assumptions of this theorem, we have $(X + Y)Z = 0$. On the other hand,

$$Z^2 = \begin{pmatrix} A^\pi BS^\pi C & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and } Z^3 = 0. \tag{2.10}$$

By Lemma 1.3 (ii)

$$UM^DU^{-1} = (X + Y)^D + Z\left((X + Y)^D\right)^2 + Z^2\left((X + Y)^D\right)^3. \tag{2.11}$$

Now, it is clear that $YX = 0$. Since AA^π is nilpotent of index r, Y is also nilpotent of index r. By Lemma 1.3 (i),

$$(X + Y)^D = \sum_{i=0}^{r-1} (X^D)^{i+1} Y^i.$$

Moreover, since $YX^D = 0$, we get

$$[(X + Y)^D]^k = \sum_{i=0}^{r-1} (X^D)^{i+k} Y^i \quad \forall\, k \geqslant 1. \tag{2.12}$$

By substituting (2.12) in (2.11) we get

$$M^D = U^{-1}\left[I + ZX^D + Z^2(X^D)^2\right] X^D \left[I + \sum_{i=1}^{r-1} (X^D)^i Y^i\right] U. \tag{2.13}$$

To obtain X^D we will apply Corollary 2.3. Since $W = I + A^DBS^\pi CA^D$ is nonsingular, using Proposition 1.1 (c), we can easily prove that

$$(WA^2A^D)^D = A^DW^{-1}, \quad (WA^2A^D)^\pi = A^\pi \text{ and } \operatorname{ind}(WA^2A^D) = 1.$$

We note that the Schur complement of WA^2A^Din X is equal to S and the conditions in Corollary 2.3 hold for this partitioned matrix. Thus,

$$X^D = \begin{pmatrix} A^DW^{-1} + A^DW^{-1}A^DBS^\sharp CAA^D & -A^DW^{-1}A^DBSS^\sharp \\ -(S^\sharp)^2CAA^D & S^\sharp \end{pmatrix}. \tag{2.14}$$

With U and U^{-1} defined as in (2.1) and R defined as in (2.8), using (2.14), (2.9) and (2.10) we compute

$$\begin{aligned} U^{-1}(X^D)^iU &= R^i \begin{pmatrix} I - A^DBS^\sharp CA^\pi & A^DBS^\pi \\ S^\sharp CA^\pi & I \end{pmatrix}, \quad \forall i \geqslant 1, \\ U^{-1}Y^iU &= \begin{pmatrix} A^\pi A^i & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall i \geqslant 1, \\ U^{-1}ZU &= \begin{pmatrix} -A^DBS^\pi C & A^\pi B - A^DBS^\pi CA^DB \\ S^\pi C & S^\pi CA^DB \end{pmatrix}, \\ U^{-1}Z^2U &= \begin{pmatrix} A^\pi BS^\pi C & A^\pi BS^\pi CA^\pi B \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

By substituting the above computations in (2.13) and rearranging terms we obtain the formula (2.7). □

We can now state the following corollary, which recovers the result of [8, Theorem 3.1] for the case when S is nonsingular.

Corollary 2.7. *Let M be partitioned as in (1.3) and let S be as in (1.4) with $\operatorname{ind}(S) \leqslant 1$. If $AA^\pi B = 0$, $CA^\pi B = 0$ and $S^\pi C = 0$, then*

$$M^D = \left[I + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix} R\right] R \left[\begin{pmatrix} I & A^D BS^\pi \\ 0 & I \end{pmatrix} + \sum_{i=0}^{r-1} R^{i+1} \begin{pmatrix} 0 & 0 \\ CA^\pi A^i & 0 \end{pmatrix}\right],$$

where $r = \operatorname{ind}(A)$ and R is defined as in (2.8) with $W = I$.

Under the assumptions of Theorem 2.6 with "$AA^\pi B = 0$, $S^\pi CA^\pi = 0$" replaced by "$CAA^\pi = 0$, $A^\pi BS^\pi = 0$", and using the Lemma 2.4, the further result may be proved in much the same way as Theorem 2.6.

Theorem 2.8. *Let M be partitioned as in (1.3) and let S be as in (1.4) with $\operatorname{ind}(S) \leqslant 1$. If $CAA^\pi = 0$, $CA^\pi B = 0$, $A^\pi BS^\pi = 0$ and $W = I + A^D BS^\pi CA^D$ is nonsingular, then*

$$M^D = \left[\begin{pmatrix} I & 0 \\ S^\pi CA^D & I \end{pmatrix} + \sum_{i=0}^{r-1} \begin{pmatrix} 0 & A^i A^\pi B \\ 0 & 0 \end{pmatrix} R^{i+1}\right] R \\ \times \left[I + R \begin{pmatrix} -BS^\pi CA^D & BS^\pi \\ C(A^\pi + (I - W)) & CA^D BS^\pi \end{pmatrix} + R^2 \begin{pmatrix} BS^\pi CA^\pi & 0 \\ CA^D BS^\pi CA^\pi & 0 \end{pmatrix}\right],$$

where $r = \operatorname{ind}(A)$ and

$$R = \begin{pmatrix} W^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A^D + A^D BS^\sharp CA^D & -A^D BS^\sharp \\ -S^\sharp CA^D & S^\sharp \end{pmatrix}. \tag{2.15}$$

Corollary 2.9. *Let M be partitioned as in (1.3) and let S be as in (1.4) with $\operatorname{ind}(S) \leqslant 1$. If $CAA^\pi = 0$, $CA^\pi B = 0$ and $BS^\pi = 0$, then*

$$M^D = \left[\begin{pmatrix} I & 0 \\ S^\pi CA^D & I \end{pmatrix} + \sum_{i=0}^{r-1} \begin{pmatrix} 0 & A^i A^\pi B \\ 0 & 0 \end{pmatrix} R^{i+1}\right] R \left[I + R \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}\right],$$

where $r = \operatorname{ind}(A)$ and R is defined as in (2.15) with $W = I$.

3. Group invertibility of partitioned matrices under block-rank condition

In [14], necessary and sufficient conditions for the existence of an inner inverse A^- such that $\operatorname{rank}(M) = \operatorname{rank}(A) + \operatorname{rank}(D - CA^- B)$ were given. When we focus attention on the Drazin inverse, it seems natural to consider the block-rank condition $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$. Here we establish a characterization for further use.

Theorem 3.1. *Let M be partitioned as in (1.3) and let S be as in (1.4). Then $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$ if and only if the following conditions hold*

$$A^\pi (A - BS^D C)A^\pi = 0, \quad A^\pi BS^\pi = 0, \quad S^\pi CA^\pi = 0 \quad \text{and} \quad SS^\pi = 0. \tag{3.1}$$

Proof. Let us introduce the matrices

$$F = \begin{pmatrix} -AA^\pi + BS^D C & B \\ C & -SS^\pi \end{pmatrix}, \quad G = \begin{pmatrix} A^2 A^D & 0 \\ 0 & S^2 S^D \end{pmatrix}.$$

Then, using Proposition 1.1(c)

$$\operatorname{rank}\begin{pmatrix} M & 0 \\ 0 & G \end{pmatrix} = \operatorname{rank}(M) + \operatorname{rank}(A^D) + \operatorname{rank}(S^D). \tag{3.2}$$

On the other hand, using that matrix multiplications by nonsingular matrices do not change the rank of the matrix we get

$$\begin{aligned}
\operatorname{rank}\begin{pmatrix} M & 0 \\ 0 & G \end{pmatrix} &= \operatorname{rank}\left(\begin{pmatrix} -I & 0 & I & 0 \\ -CA^D & I & CA^D & SS^D \\ 0 & 0 & I & 0 \\ SS^DCA^D & -SS^D & -SS^DCA^D & S^\pi \end{pmatrix}\begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & A^2A^D & 0 \\ 0 & 0 & 0 & S^2S^D \end{pmatrix}\right) \\
&= \operatorname{rank}\left(\begin{pmatrix} -A & -B & A^2A^D & 0 \\ CA^\pi & S & CAA^D & S^2S^D \\ 0 & 0 & A^2A^D & 0 \\ -SS^DCA^\pi & -S^2S^D & -SS^DCAA^D & 0 \end{pmatrix}\right. \\
&\qquad \left.\times \begin{pmatrix} I & 0 & -AA^D & 0 \\ -S^DC & -I & 0 & 0 \\ I & 0 & A^\pi & 0 \\ S^DC & I & 0 & I \end{pmatrix}\right) \\
&= \operatorname{rank}\begin{pmatrix} -AA^\pi + BS^DC & B & A^2A^D & 0 \\ C & -SS^\pi & 0 & S^2S^D \\ A^2A^D & 0 & 0 & 0 \\ 0 & S^2S^D & 0 & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} F & G \\ G & 0 \end{pmatrix}.
\end{aligned} \tag{3.3}$$

We note that $G^- = \begin{pmatrix} A^D & 0 \\ 0 & S^D \end{pmatrix}$ is an inner inverse of G and $I - GG^- = I - G^-G = \begin{pmatrix} A^\pi & 0 \\ 0 & S^\pi \end{pmatrix}$. Further,

$$(I - GG^-)F(I - G^-G) = \begin{pmatrix} -A^\pi(A - BS^DC)A^\pi & A^\pi BS^\pi \\ S^\pi CA^\pi & -SS^\pi \end{pmatrix} := N.$$

By Lemma 1.4 (ii) we obtain

$$\operatorname{rank}\begin{pmatrix} F & G \\ G & 0 \end{pmatrix} = 2\operatorname{rank}(G) + \operatorname{rank}(N) = 2\operatorname{rank}(A^D) + 2\operatorname{rank}(S^D) + \operatorname{rank}(N). \tag{3.4}$$

Using (3.2)–(3.4) we obtain

$$\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D) + \operatorname{rank}(N).$$

Hence $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$ if and only if $\operatorname{rank}(N) = 0$ or, equivalently, conditions (3.1) hold. □

Corollary 3.2. *Let M be partitioned as in (1.3) and let S be as in (1.4). If S is nonsingular, then*

$$\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S) \Leftrightarrow A^\pi(A - BS^{-1}C)A^\pi = 0.$$

Corollary 3.3. *Let M be partitioned as in (1.3) and let S be as in (1.4). Then*

$$\operatorname{rank}(M) = \operatorname{rank}(A^D) \Leftrightarrow A^\pi B = 0, CA^\pi = 0, S = 0 \text{ and } AA^\pi = 0.$$

Remark 3.4. Let $r = \operatorname{ind}(A)$. A geometrical reformulation of conditions (3.1) is as follows:

$$B\mathcal{N}(S) \cup (A - BS^DC)\mathcal{N}(A^r) \subset \mathcal{R}(A^r), \quad C\mathcal{N}(A^r) \subset \mathcal{R}(S) \quad \text{and} \quad \mathcal{R}(S) = \mathcal{R}(S^2).$$

We can now state the main result of this section.

Theorem 3.5. *Let M be partitioned as in (1.3) and let S be as in (1.4). If $CA^\pi B = 0$ and $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$, then $\operatorname{ind}(M) = 1$ if and only if $W = I + A^D BS^\pi CA^D$ is nonsingular. In this case the group inverse of M has the form*

$$M^\sharp = \left[R + \begin{pmatrix} -A^D BS^\pi C & (A^\pi + (I - W))B \\ S^\pi C & S^\pi CA^D B \end{pmatrix} R^2\right] \begin{pmatrix} I - A^D BS^\sharp CA^\pi & A^D BS^\pi \\ S^\sharp CA^\pi & I \end{pmatrix}, \tag{3.5}$$

where R is defined as in (2.8).

Proof. First, note that under assumption $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$, conditions (3.1) also hold. From $AA^\pi = A^\pi BS^D CA^\pi$ and $CA^\pi B = 0$ it follows that $AA^\pi B = 0$ and $CAA^\pi = 0$. Further, using U^{-1} and V^{-1} defined as in (2.1) and (2.5), respectively, we obtain

$$\begin{aligned} \operatorname{rank}(V^{-1}MMU^{-1}) &= \operatorname{rank}\left(\begin{pmatrix} AA^D(A - BS^\sharp CA^\pi) & BS^\pi \\ CA^\pi & S \end{pmatrix}\begin{pmatrix} (A - A^\pi BS^\sharp C)AA^D & A^\pi B \\ S^\pi C & S \end{pmatrix}\right) \\ &= \operatorname{rank}\begin{pmatrix} A^3A^D + AA^D BS^\pi CAA^D & 0 \\ 0 & S^2 \end{pmatrix} \\ &= \operatorname{rank}(A^3A^D + AA^D BS^\pi CAA^D) + \operatorname{rank}(S). \end{aligned}$$

Hence, since the rank is invariant under matrix multiplications by nonsingular matrices, $\operatorname{rank}(M^2) = \operatorname{rank}(A^D) + \operatorname{rank}(S)$ if and only if $\operatorname{rank}(A^3A^D + AA^D BS^\pi CAA^D) = \operatorname{rank}(A^D)$. The latter relation holds if and only if

$$\mathcal{R}(A^3A^D) \cap \mathcal{N}(AA^D + AA^D BS^\pi C(A^D)^2) = \{0\},$$

by Lemma 1.4 (i). Which is equivalent to the fact that $A^\pi + AA^D + AA^D BS^\pi C(A^D)^2$ is nonsingular. Hence $I + A^D BS^\pi CA^D$ is also nonsingular and we conclude the first part of the proof.

By applying Theorem 2.6 to this case, we get (3.5) and the proof is finished. □

Here we give two important consequences.

Corollary 3.6. *Let M be partitioned as in (1.3) and let S be as in (1.4). If A is nonsingular and $\operatorname{rank}(M) = \operatorname{rank}(A) + \operatorname{rank}(S^D)$, then $\operatorname{ind}(M) = 1$ if and only if $A^2 + BS^\pi C$ is nonsingular. In this case*

$$M^\sharp = \left[R + \begin{pmatrix} -A^{-1}BS^\pi C & -A^{-1}BS^\pi CA^{-1}B \\ S^\pi C & S^\pi CA^{-1}B \end{pmatrix} R^2\right] \begin{pmatrix} I & A^{-1}BS^\pi \\ 0 & I \end{pmatrix},$$

where R is defined as in (2.8) with $A^D = A^{-1}$.

Corollary 3.7. *Let M be partitioned as in (1.3). If $\operatorname{rank}(M) = \operatorname{rank}(A^D)$, then $\operatorname{ind}(M) = 1$ if and only if $W = I + A^D BCA^D$ is nonsingular. In this case,*

$$M^\sharp = \begin{pmatrix} I \\ CA^D \end{pmatrix} A \left[A^D W^{-1}\right]^2 \begin{pmatrix} I & A^D B \end{pmatrix}. \tag{3.6}$$

Proof. From Theorem 2.6 we get

$$M^\sharp = \left[\begin{pmatrix} A^D W^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -A^D BC & (I - W)B \\ C & CA^D B \end{pmatrix}\begin{pmatrix} (A^D W^{-1})^2 & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} I & A^D B \\ 0 & I \end{pmatrix},$$

which can be rewritten as (3.6). □

Next, we explore the group invertibility of M when the block-rank condition and $BS^{\pi}C = 0$ hold. Further, we will derive a characterization in the case when S is nonsingular.

Theorem 3.8. *Let M be partitioned as in (1.3) and let S be as in (1.4). If $BS^{\pi}C = 0$ and $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$, then $\operatorname{ind}(M) = 1$ if and only if $W = I + Z(I + C(A^D)^2BSS^{\sharp})$ is nonsingular, where $Z = S^{\sharp}CA^{\pi}BS^{\sharp}$. In this case,*

$$M^{\sharp} = \left[I + \begin{pmatrix} 0 & (I + A^D BS^{\sharp}C)A^{\pi}B \\ S^{\pi}C & S^{\pi}CA^DB - S^{\sharp}CA^{\pi}B \end{pmatrix} RT \right] RT, \tag{3.7}$$

where

$$R = \begin{pmatrix} A^D + A^DBS^{\sharp}CA^D & -A^DBS^{\sharp} \\ -S^{\sharp}CA^D & S^{\sharp} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & W^{-1} \end{pmatrix}, \tag{3.8}$$

$$T = \begin{pmatrix} I - A^DBW^{-1}(S^{\sharp}CA^{\pi} + ZCA^D) & A^DB(I - W^{-1}(SS^{\sharp} + ZC(A^D)^2B)) \\ S^{\sharp}CA^{\pi} + ZCA^D & I + ZC(A^D)^2B \end{pmatrix}. \tag{3.9}$$

Proof. Let U^{-1} and V^{-1} be defined as in (2.1) and (2.5), respectively. Using $AA^{\pi} = A^{\pi}BS^{\sharp}CA^{\pi}, BS^{\pi}C = 0$ and $SS^{\pi} = 0$, applying Lemma 1.4 (iii) we get

$$\begin{aligned} \operatorname{rank}(M^2) &= \operatorname{rank}\left(\begin{pmatrix} I & AA^DBS^{\sharp} \\ 0 & I \end{pmatrix} V^{-1}MMU^{-1} \right) \\ &= \operatorname{rank} \begin{pmatrix} A^3A^D & AA^DBS \\ -CA^{\pi}BS^{\sharp}CAA^D & S^2 + CA^{\pi}B \end{pmatrix} \\ &= \operatorname{rank}(A^3A^D) + \operatorname{rank} \begin{pmatrix} 0 & 0 \\ 0 & S^2 + CA^{\pi}B + CA^{\pi}BS^{\sharp}C(A^D)^2BS \end{pmatrix} \\ &= \operatorname{rank}(A^D) + \operatorname{rank}(S^2 + CA^{\pi}B + CA^{\pi}BS^{\sharp}C(A^D)^2BS). \end{aligned}$$

Hence, on account that $A^{\pi}BS^{\pi} = 0$ and by Lemma 1.4 (i), it follows that $\operatorname{rank}(M^2) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$ if and only if

$$\dim \mathcal{R}(S) \cap \mathcal{N}(S + CA^{\pi}BS^{\sharp} + CA^{\pi}BS^{\sharp}C(A^D)^2BSS^{\sharp}) = 0,$$

which is equivalent to the fact that $(S + S^{\pi})(I + S^{\sharp}CA^{\pi}BS^{\sharp}(I + C(A^D)^2BSS^{\sharp}))$ is nonsingular. Hence, since $S + S^{\pi}$ is nonsingular, we conclude the first part of the proof.

Let U and U^{-1} be defined as in (2.1). Using $AA^{\pi} = A^{\pi}BS^{\sharp}CA^{\pi}$ and $BS^{\pi}C = 0$, we get

$$UMU^{-1} := X + Y,$$

where

$$X = \begin{pmatrix} A^2A^D & A^DBS \\ S^{\sharp}C(A^2A^D - A^{\pi}BS^{\sharp}CAA^D) & S + S^{\sharp}CA^{\pi}B + S^{\sharp}CA^DBS \end{pmatrix},$$

$$Y = \begin{pmatrix} -A^{\pi}BS^{\sharp}CAA^D & A^{\pi}B \\ S^{\pi}C & 0 \end{pmatrix}.$$

We note that $XY = 0$ and $Y^2 = 0$. By Lemma 1.3 (ii),

$$M^{\sharp} = U^{-1}(X^D + Y(X^D)^2)U. \tag{3.10}$$

To get a representation of X^D we will apply Corollary 2.3. First, we observe that the Schur complement of A^2A^D in X is equal to WS. Moreover, we can easily see that

$$(WS)^D = S^{\sharp}W^{-1}, \quad (WS)^{\pi} = S^{\pi}, \quad \operatorname{ind}(WS) = 1.$$

Since $\operatorname{ind}(A^2A^D) = 1$ also holds, applying Corollary 2.3 to the partitioned matrix X we conclude

$$X^D = \begin{pmatrix} A^D + (A^D)^2BW^{-1}S^\sharp C(AA^D - A^\pi BS^\sharp CA^D) & -(A^D)^2BSS^\sharp W^{-1} \\ -S^\sharp W^{-1}S^\sharp C(AA^D - A^\pi BS^\sharp CA^D) & S^\sharp W^{-1} \end{pmatrix}. \tag{3.11}$$

Using the notations (3.8) and (3.9), after some computations we get $U^{-1}X^DU = RT$ and

$$U^{-1}YU = \begin{pmatrix} (I + A^DBS^\sharp C)AA^\pi & (I + A^DBS^\sharp C)A^\pi B \\ S^\pi C - S^\sharp CAA^\pi & S^\pi CA^DB - S^\sharp CA^\pi B \end{pmatrix}.$$

Finally, by substituting the above expressions in (3.10) we get (3.7). □

Corollary 3.9. *Let M be partitioned as in (1.3) and let S be as in (1.4). If S is nonsingular and $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S)$, then $\operatorname{ind}(M) = 1$ if and only if the matrix $S^2 + CA^\pi B(I + C(A^D)^2BS)$ is nonsingular. In this case $M^\sharp$ is given by (3.7) with $S^\sharp = S^{-1}$ and $S^\pi = 0$.*

We finish with a related result for the Drazin inverse.

Theorem 3.10. *Let M be partitioned as in (1.3) and let S be as in (1.4). If $\operatorname{rank}(M) = \operatorname{rank}(A^D) + \operatorname{rank}(S^D)$, $A^DBS = 0$ and $SCA^D = 0$ hold, then*

$$M^D = \begin{pmatrix} I \\ CA^D \end{pmatrix} A[(WA)^D]^2 \begin{pmatrix} I & A^DB \end{pmatrix} + \begin{pmatrix} A^\pi BS^\sharp \\ I \end{pmatrix} S[(\widetilde{W}S)^D]^2 \begin{pmatrix} S^\sharp CA^\pi & I \end{pmatrix},$$

where $W = AA^D + A^DBCA^D$ and $\widetilde{W} = SS^\sharp + S^\sharp CA^\pi BS^\sharp$.

Proof. We split $M = \begin{pmatrix} A^2A^D & AA^DB \\ CAA^D & CA^DB \end{pmatrix} + \begin{pmatrix} AA^\pi & A^\pi B \\ CA^\pi & S \end{pmatrix} := X + Y$.

From $S^\pi CA^\pi = 0$ and $SCA^D = 0$ we obtain $CA^\pi = SS^DC$. Using this latter relation and $A^DBS = 0$ we get $XY = 0$. Analogously we can see that $YX = 0$. Then, $M^D = X^D + Y^D$. We can apply (1.7) to obtain the Drazin inverse of X,

$$X^D = \begin{pmatrix} I \\ CA^D \end{pmatrix} A[(WA)^D]^2 \begin{pmatrix} I & A^DB \end{pmatrix}, \quad W = AA^D + A^DBCA^D.$$

On the other hand, the generalized Schur complement of S in Y, given by $AA^\pi - A^\pi BS^DCA^\pi$, is equal to zero. On account that $A^\pi BS^\pi = 0$ and $S^\pi CA^\pi = 0$, we can apply the symmetrical result of (1.7) to obtain

$$Y^D = \begin{pmatrix} A^\pi BS^\sharp \\ I \end{pmatrix} [S(\widetilde{W}S)^D]^2 \begin{pmatrix} S^\sharp CA^\pi & I \end{pmatrix}, \quad \widetilde{W} = SS^\sharp + S^\sharp CA^\pi BS^\sharp.$$

This completes the proof. □

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